

# SOBOLEV AND SCHWARTZ: TWO FATES AND TWO FAMES

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*On the Occasion of the Centenary of  
the Birth of S. L. Sobolev*

ABSTRACT. This is a brief overview of the lives and contributions of S. L. Sobolev and L. Schwartz, the cofounders of distribution theory.

In the history of mathematics there are quite a few persons whom we prefer to recollect in pairs. Listed among them are Euclid and Diophant, Newton and Leibniz, Bolyai and Lobachevskiĭ, Hilbert and Poincaré, as well as Bourbaki and Arnold. In this series we enroll Sobolev and Schwartz who are inseparable from one of the most brilliant discoveries of the twentieth century, the theory of generalized functions or distribution theory, providing a revolutionary new approach to partial differential equations.

The most vibrant and lasting achievements of mathematics reside in formulas and lists. There are pivotal distinctions between lists and formulas. The former deposit that which was open for us. The lists of platonic solids, elementary catastrophes, and finite simple groups are next of kin to the *Almagest* and herbaria. They are the objects of admiration, tremendous and awe-struck. The article of the craft of mathematics is a formula. Each formula enters into life as an instance of materialization of mathematical creativity. No formula serves only the purpose it was intended to. In part, any formula is reminiscent of household appliances, toys, or software. It is a very rare event that somebody reads the user's guide of a new TV set or the manual for running a new computer program. Usually everyone utilizes his or her new gadgets experimentally by pressing whatever keys and switches. In much the same way we handle formulas. We painstakingly "twist and turn" them, audaciously insert new parameters, willfully interpret symbols, and so on.

MATHEMATICS IS THE CRAFT OF FORMULAS AND THE ART OF CALCULUS. If someone considers this claim as feeble and incomplete, to remind is in order that, logically speaking, set theory is just an instance of the first order predicate calculus.

Distribution theory has become the calculus of today. Of such a scale and scope is the scientific discovery by Sobolev and Schwartz.

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Partly printed in [1] with unauthorized omissions.

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## 1. SERGEĬ L'VOVICH SOBOLEV

Sobolev was born in St. Petersburg on October 6, 1908 in the family of Lev Aleksandrovich Sobolev, a solicitor. Sobolev's grandfather on his father's side descended from a family of Siberian Cossacks.

Sobolev was bereaved of his father in the early childhood and was raised by his mother Natal'ya Georgievna who was a highly-educated teacher of literature and history. His mother also had the second speciality: she graduated from a medical institute and worked as a tutor at the First Leningrad Medical Institute. She cultivated in Sobolev the decency, indefatigability, and endurance that characterized him as a scholar and personality.

Sobolev fulfilled the program of secondary school at home, revealing his great attraction to mathematics. During the Civil War he and his mother lived in Kharkov. When living there, he studied at the preparatory courses of a evening technical school for one semester. At the age of 15 he completed the obligatory programs of secondary school in mathematics, physics, chemistry, and other natural sciences, read the classical pieces of the Russian and world literature as well as many books on philosophy, medicine, and biology.

After the family had transferred from Kharkov to Petersburg in 1923, Sobolev entered the graduate class of School No. 190 and finished with honors in 1924, continuing his study at the First State Art School in the piano class. At the same year he entered the Faculty of Physics and Mathematics of Leningrad State University (LSU) and attended the lectures of Professors N. M. Günter, V. I. Smirnov, G. M. Fikhtengolts, and others. He made his diploma on the analytic solutions of a system of differential equations with two independent variables under the supervision of Günter.

Günter propounded the idea that the set functions are inevitable in abstracting the concept of solution to a differential equation. Günter's approach influenced the further train of thought of Sobolev.<sup>1</sup>

After graduation from LSU in 1929, Sobolev started his work at the Theoretical Department of the Leningrad Seismological Institute. In a close cooperation with Smirnov he then solved some fundamental problems of wave propagation. It was Smirnov whom Sobolev called his teacher alongside Günter up to his terminal days.

Since 1932 Sobolev worked at the Steklov Mathematical Institute in Leningrad; and since 1934, in Moscow. He continued the study of hyperbolic equations and proposed a new method for solving the Cauchy problem for a hyperbolic equation with variable coefficients. This method was based on a generalization of the Kirchhoff formula. Research into hyperbolic equations led Sobolev to revising the classical concept of a solution to a differential equation. The concept of a generalized or weak solution of a differential equation was considered earlier. However, it was exactly in the works by Sobolev that this concept was elaborated and applied systematically. Sobolev posed and solved the Cauchy problem in spaces of functionals, which was based on the revolutionary extension of the Eulerian concept of function and declared 1935 as the date of the birth of the theory of distributions.

Suggesting his definition of generalized derivative, Sobolev enriched mathematics with the spaces of functions whose weak derivatives are integrable to some power. These are now called *Sobolev spaces*.

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<sup>1</sup>It was A. M. Vershik and V. I. Arnold who attracted the author's attention to the especial role of Günter in the prehistory of distribution theory.

Let  $f$  and  $g$  be locally summable functions on an open subset  $G$  of  $\mathbb{R}^n$ , and let  $\alpha$  be a multi-index. A function  $g$ , denoted by  $D^\alpha f$ , is the *generalized derivative in the Sobolev sense* or *weak derivative* of  $f$  of order  $\alpha$  provided that

$$\int_G f(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_G g(x) \varphi(x) dx,$$

for every *test function*  $\varphi$ , i. e. such that the support of  $\varphi$  is a compact subset of  $G$  and  $\varphi$  is  $|\alpha| = \alpha_1 + \dots + \alpha_n$  times continuously differentiable in  $G$ , where  $D^\alpha \varphi$  is the classical derivative of  $\varphi$  of order  $\alpha$ . The vector space  $W_p^l$ , with  $p \geq 1$ , of the (cosets of) locally summable  $f$  on  $G$ , whose all weak derivatives  $D^\alpha f$  with  $|\alpha| \leq l$  are  $p$ -summable in  $G$  becomes a Banach space under the norm:

$$\|f\|_{W_p^l} = \left( \int_G |f|^p dx \right)^{1/p} + \sum_{|\alpha| \leq l} \left( \int_G |D^\alpha f|^p dx \right)^{1/p}.$$

Sobolev found the general criteria for equivalence of various norms on  $W_p^l$  and showed that these spaces are the natural environment for posing the boundary value problems for elliptic equations. This conclusion was based on his thorough study of the properties of Sobolev spaces. The most important facts are *embedding theorems*. Each embedding theorem estimates the operator norm of an embedding, yielding special inequalities between the norms of one and the same function inside various spaces.

The contributions of Sobolev brought him recognition in the USSR. In 1933 Sobolev was elected a corresponding member of the Academy of Sciences at the age of 24 years. In 1939 he became a full member of the Academy and remained the youngest academician for many years.

Inspired by military applications in the 1940s, Sobolev studying the system of differential equations describing small oscillations of a rotating fluid. He obtained the conditions for stability of a rotating body with a filled-in cavity which depend on the shape and parameters of the cavity. Moreover, he elaborated the cases in which the cavity is a cylinder or an ellipsoid of rotation. This research by Sobolev signposted another area of the general theory which concerns the Cauchy and boundary value problems for the equations and systems that are not solved with respect to higher time derivatives.

In the grievous years of the Second World War from 1941 to 1944 Sobolev occupied the position of the director of the Steklov Mathematical Institute.

Sobolev was one of the first scientists who foresaw the future of computational mathematics and cybernetics. From 1952 to 1960 he held the chair of the first national department of computational mathematics at Moscow State University. This department has played a key role in the development of many important areas of the today's mathematics.

Addressing the problems of computational mathematics, Sobolev lavishly applied the apparatus of the modern sections of the theoretical core of mathematics. It is typical for him to pose the problems of computational mathematics within functional analysis. Winged are his words that “to conceive the theory of computations without Banach spaces is impossible just as trying to conceive it without computers.”

It is worthwhile to emphasize the great role in the uprise of cybernetics and other new areas of research in this country which was played by the publications

and speeches of Sobolev who valiantly defended the new trends in science from the ideologized obscurantism.

To overrate is difficult the contribution of Sobolev to the design of the nuclear shield of this country. From the first stages of the atomic project of the USSR he was listed among the top officials of Laboratory No. 2 which was renamed for secrecy reasons into the Laboratory of Measuring Instruments (abbreviated as LI-PAN in Russian). Now LIPAN lives as the Kurchatov Center. The main task of the joint work with I. K. Kikoin was the implementation of gaseous diffusive uranium enrichment for creation of a nuclear explosive device.

Sobolev administered and supervised various computational teams, studied the control of the industrial processes of isotope separation, struggled for the low costs of production and made decisions on many managerial and technological matters. For his contribution to the A-bomb project, Sobolev twice gained a Stalin Prize of the First Degree. In January of 1952 Sobolev was awarded with the highest title of the USSR: he was declared the Hero of the Socialist Labor for exceptional service to the state.

Sobolev's research was inseparable from his management in science. At the end of the 1950s M. A. Lavrent'ev, S. L. Sobolev, and S. A. Khristianovich came out with the initiative to organize a new big scientific center, the Siberian Division of the Academy of Sciences. For many scientists of the first enrolment to the Siberian Division it was the example of Sobolev, his authority in science, and the attraction of his personality that yielded the final argument in deciding to move to Novosibirsk.

The Siberian period of Sobolev's life in science was marked by the great achievements in the theory of cubature formulas. Approximate integration is one of the main problems in the theory of computations—the cost of computation of multi-dimensional integrals is extremely high. Optimizing the formulas of integration is understood now to be the problem of minimizing the norm of the error on some function space. Sobolev suggested new approaches to the problem and discovered marvelous classes of optimal cubature formulas.

Sobolev merits brought him many decorations and signs of distinction. In 1988 he was awarded the highest prize of the Russian Academy of Sciences, the Lomonosov Gold Medal.

Sobolev passed away in Moscow on January 3, 1989.

## 2. LAURENT SCHWARTZ

Schwartz was born in Paris on March 5, 1915 in the family of Anselme Schwartz, a surgeon. There were quite a few prominent persons among his next of kin. J. Hadamard was his granduncle. Many celebrities are listed in the line of his mother's line Claire Debrés: Several Gaullist politicians belonged to the Debérs. In 1938 Schwartz married Marie-Hélène Lévy, the daughter of the outstanding mathematician P. Lévy who was one of the forefathers of functional analysis. Marie-Hélène had become a professional mathematician and gained the position of a full professor in 1963.

The munificent gift of Schwartz was revealed in his lycée years. He won the most prestigious competition for high school students, Concours Général in Latin. Schwartz was unsure about his future career, hovering between geometry and “classics” (Greek and Latin). It is curious that Hadamard had a low opinion of Schwartz mathematical plans, since the sixteen-years old Laurent did not know the Riemann

zeta function. By a startling contrast, Schwartz was boosted to geometry by the pediatrician Robert Debré and one of his teachers of classics.

In 1934 Schwartz passed examinations to the École Normale Supérieure (ENS) after two years of preparation. He was admitted together with Gustave Choquet, a winner of the Concours Général in mathematics, and Marie-Hélène, one of the first females in the ENS. The mathematical atmosphere of those years in the ENS was determined by É. Borel, É. Cartan, A. Denjoy, M. Fréchet, and P. Montel. The staff of the neighboring Collège de France included H. Lebesgue who delivered lectures and Hadamard who conducted seminars. It was in his student years that Schwartz had acquired the irretrievable and permanent love to probability theory which grew from conversations with his future father-in-law Lévy.

After graduation from the ENS Schwartz decided to be drafted in the compulsory military service for two years. He had to stay in the army in 1939–1940 in view of the war times. These years were especially hard for the young couple of the Schwartzes. It was unreasonable for Jews to stay in the occupied zone. The Schwartzes had to escape from the native north and manage to survive on some modest financial support that was offered in particular by Michelin, a world-renowned tire company. In 1941 Schwartz was in Toulouse for a short time and met H. Cartan and J. Delsarte who suggested that the young couple should move to Clermont-Ferrand, the place of temporary residence of the group of professors of Strasbourg University which had migrated from the German occupation. These were J. Dieudonné, Ch. Ehresmann, A. Lichnerowicz, and S. Mandelbrojt. In Clermont-Ferrand Schwartz completed his Ph. D. Thesis on approximation of a real function on the axis by sums of exponentials.

Unfortunately, the war had intervened into the mathematical fate of Schwartz. His family had to change places under false identities. Curiously, in the time of the invention of distributions in November of 1944 Schwartz used the identity of Seli-martin. The basics of Schwartz's theory appeared in the *Annales* of the University of Grenoble in 1945.<sup>2</sup> Schwartz described the process of invention as “cerebral percolation.” After a year's stay in Grenoble, Schwartz acquired a position in Nansy, plunging to the center of “Bourbakism.” It is well known that N. Bourbaki resided in Nancago, a mixture of Nancy and Chicago. A. Weyl lived in Chicago, while Delsarte and Dieudonné were in Nancy. Before long Schwartz was enrolled in the group of Bourbaki. In 1950 he was awarded with the Fields medal for distribution theory. His now-celebrated two-volume set *Théorie des Distributiones* was printed a short time later.

In 1952 Schwartz returned back to Paris and began lecturing in Sorbonne; and since 1959, in the École Polytechnique in company with his father-in-law Lévy. Many celebrities were the direct students of Schwartz. Among them we list A. Grothendieck, J.-L. Lions, B. Malgrange, and A. Martineau.

Schwartz wrote: “To discover something in mathematics is to overcome an inhibition and a tradition. You cannot move forward if you are not subversive.” This statement is in good agreement with the very active and versatile public life of Schwartz. He joined Trotskists in his green years, protesting against the monstrosities of capitalism and Stalin's terror of the 1930s. Since then he had never agreed with anything that he viewed as violation of human rights, oppression, or injustice. He was very active in struggling against the American war in Vietnam and the

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<sup>2</sup>Cp. [8].

Soviet invasion in Afghanistan. He fought for liberation of a few mathematicians that were persecuted for political reasons, among them Jose Luis Massera, Vaclav Benda, et al.

Schwartz was an outstanding lepidopterist and had collected more than 20 000 butterflies. It is not by chance that the butterflies are depicted on the soft covers of the second edition of his *Théorie des Distributiones*.

Schwartz passed away in Paris on June 4, 2002.

### 3. ADVANCES OF DISTRIBUTION THEORY

Distribution theory stems from the intention to apply the technologies of functional analysis to studying partial differential equations. Functional analysis rests on algebraization, geometrization, and socialization of analytical problems. By socialization we usually mean the inclusion of a particular problem in an appropriate class of its congeners. Socialization enables us to erase the “random features,”<sup>3</sup> eliminating the difficulties of the insurmountable specifics of a particular problem. In the early 1930s the merits of functional analysis were already demonstrated in the area of integral equations. The time was ripe for the differential equations to be placed on the agenda.

It is worth observing that the contemplations about the nature of integration and differentiation underlie most of the theories of the present-day functional analysis. This is no wonder at all in view of the key roles of these remarkable linear operations. Everyone knows that integration possesses a few more attractive features than differentiation: the integral is monotone and raises smoothness. Derivation lacks these nice properties completely. Everyone knows as well that the classical derivative yields a closed yet unbounded operator (with respect to the natural uniform convergence topology that is induced by the Chebyshev sup-norm). The series of smooth functions cannot be differentiated termwise in general, which diminishes the scope of applications of analysis to differential equations.

There is practically no denying today that the concept of generalized derivative occupies a central place in distribution theory. Derivation is now treated as the operator that acts on the nonsmooth functions according to the same integral laws as the procedure of taking the classical derivative. It is exactly this approach that was pursued steadily by Sobolev. The new turnpike led to the stock of previously impossible differentiation formulas. It turned out that each distribution possesses derivatives of all orders, every series of distributions may be differentiated termwise however often, and many “traditionally divergent” Fourier series admit presentations by explicit formulas. Mathematics has acquired additional fantastic degrees of freedom, which makes immortal the name of Sobolev as a pioneer of the calculus of the twentieth century.

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<sup>3</sup>This is a cliché with a century-old history. The famous Russian symbolist Alexander Blok (1880–1921) used the concept of random feature in his incomplete poem “Revenge” as of 1910 [7, p. 482]. The prologue of this poem contains the lines that are roughly rendered in English as follows:

You share the gift of prudent measure  
 For what keen vision might perceive.  
 Erasing random features, treasure  
 The world of beauty to receive.

The detailed expositions of the new theory by Sobolev and Schwartz had appeared practically at the same time. In 1950 the first volume of *Théorie des Distributions* was published in Paris, while Sobolev's book *Applications of Functional Analysis in Mathematical Physics* was printed in Leningrad. In 1962 the Siberian Division of the Academy of Sciences of the USSR reprinted the book, while in 1963 it was translated into English by the American Mathematical Society. The second edition of the Schwartz book was published in 1966, slightly enriched with a generalized version of the de Rham currents. Curiously, Schwartz left the historical overview practically the same as in the introduction to the first edition.

The new methods of distribution theory turned out so powerful as to enable mathematicians to write down, in explicit form, the general solution of an arbitrary partial differential equation with constant coefficients. In fact, everything reduces to existence of fundamental solutions; i. e., to the case of the Dirac delta-function on the right-hand side of the equation under consideration. The existence of these solutions was already established in 1953 and 1954 by B. Malgrange and L. Ehrenpreis independently of each other. However, it was only in 1994 that some formula for a fundamental solution was exhibited by H. König. Somewhat later N. Ortner and P. Wagner found a more elementary formula. Their main result is as follows:<sup>4</sup>

**Theorem.** Assume that  $P(\partial) \in \mathbb{C}[\partial]$ , where  $P$  is a polynomial of degree  $m$ . Assume further that  $\eta \in \mathbb{R}^n$  and  $P_m(\eta) \neq 0$ , where  $P_m$  is the principal part of  $P$ ; i. e.,  $P_m = \sum_{|\alpha|=m} a_\alpha \partial^\alpha$ . Then the distribution  $E$  given as

$$E := \frac{1}{P_m(\eta)} \int_{\mathbb{T}} \lambda^m e^{\lambda \eta x} \mathfrak{F}_{\xi \rightarrow x}^{-1} \left( \frac{\overline{P(i\xi + \lambda \eta)}}{P(i\xi + \lambda \eta)} \right) \frac{d\lambda}{2\pi i \lambda}$$

is a fundamental solution of the operator  $P(\partial)$ . Moreover,  $E/\cosh(\eta x) \in \mathcal{S}'(\mathbb{R}^n)$ .

It stands to reason to inspect the structure of the formula which reveals the role of the distributional Fourier transform  $\mathfrak{F}$  and the Schwartz space  $\mathcal{S}'(\mathbb{R}^n)$  comprising tempered distributions.<sup>5</sup>

The existence of a fundamental solution of an arbitrary partial differential equation with constant coefficients is reverently called the *Malgrange–Ehrenpreis Theorem*. It is hard to overestimate this splendid achievement which remains one of the splendid triumphs of the abstract theory of topological vector spaces.

The road from solutions in distributions to standard solutions lies through Sobolev spaces. Study of the embeddings and traces of Sobolev spaces has become one of the main sections of the modern theory of real functions. Suffice it to mention S. M. Nikol'skiĭ, O. V. Besov, G. Weiss, V. P. Il'in, and V. G. Mazya in order to conceive the greatness of this area of mathematical research. The titles of dozen books mention Sobolev spaces, which is far from typical in the present-day science.

The broad stratum of modern studies deals with applications of distributions in mathematical and theoretical physics, complex analysis, the theory of pseudodifferential operators, Tauberian theorems, and other sections of mathematics.

The physical sources of distribution theory, as well as the ties of the latter with theoretical physics, are the topics of paramount importance. They require a special

<sup>4</sup>Cp. [12] and [13, Theorem 2.3].

<sup>5</sup>Also known as “generalized functions of slow growth.”

scrutiny that falls beyond the scope of this article.<sup>6</sup> We will confine exposition to the concise historical comments by V. S. Vladimirov:<sup>7</sup>

It was already the creators of this theory, S. L. Sobolev [5] and L. Schwartz [19] who studied the applications of the theory of generalized functions in mathematical physics. After a conversation with S. L. Sobolev about generalized functions, N. N. Bogolyubov used the Sobolev classes [3] of test and generalized functions  $C_{\text{comp}}^m$  and  $(C_{\text{comp}}^m)^*$  in constructing his axiomatic quantum field theory [20]–[22]. The same related to the Wightman axiomatics [23]. Moreover, it is impossible in principle to construct any axiomatics of quantum field theory without generalized functions. Furthermore, in the theory of the dispersion relations [24] that are derived from the Bogolyubov axiomatics, the generalized functions, as well as their generalizations—hyperfunctions, appear as the boundary values of holomorphic functions of (many) complex variables. This fact, together with the relevant aspects such as Bogolyubov’s “Edge-of-the-Wedge” Theorem, essentially enriches the theory of generalized functions.

#### 4. VARIOUS OPINIONS ABOUT THE HISTORY OF DISTRIBUTIONS

J. Leray was one of the most prominent French mathematicians of the twentieth century. He was awarded with the Lomonosov Gold Medal together with Sobolev in 1988. Reviewing the contributions of Sobolev from 1930 to 1955 in the course of Sobolev’s election to the Academy of Sciences of the Institute of France in 1967, J. Leray wrote:<sup>8</sup>

Distribution theory is now well developed due to the theory of topological vector spaces and their duality as well as the concept of tempered distribution which is one of the important achievements of L. Schwartz (Paris) which enabled him to construct the beautiful theory of the Fourier transform for distributions; G. de Rham supplied the concept of distribution with that of current which comprises the concepts of differential form and topological chain; L. Hörmander (Lund, Princeton), B. Malgrange (Paris), J.-L. Lions (Paris) used the theory of distributions to renew the theory of partial differential equations; while P. Lelong (Paris) established one of the fundamental properties of analytic sets. The comprehensive two-volume treatise by L. Schwartz and even more comprehensive five-volume treatise<sup>9</sup> by Gelfand and Shilov (Moscow) are the achievements of so great an importance that even the French contribution deserves the highest awards of our community. The applications of distribution theory in all areas of mathematics, theoretical physics, and numerical analysis remind of the dense forest hiding the tree whose seeds it has grown from. However, we know that if Sobolev had fail to make his discovery about 1935 in Russia, it would be committed in France by 1950 and somewhat later in Poland; the USA also flatters itself that they would

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<sup>6</sup>Some historical details are collected in [26]. Also see [27]. J.-M. Kantor kindly made his article available to the author before publication with a courteous cooperation of Ch. Davis, Editor-in-Chief of *The Mathematical Intelligencer*. It was the proposal of Ch. Davis that the article by J.-M. Kantor be supplemented with the short comments [28] and [29].

<sup>7</sup>Cited from the handwritten review for the *Herald of the Russian Academy of Sciences*, dated as of December 10, 2003.

<sup>8</sup>Cp. [14].

<sup>9</sup>In fact, the series consists of 6 volumes.



make this discovery in the same years: The science and art of mathematics would be late only by 15 years as compared with Russia. . . .

In sharp contrast with this appraisal, we cite F. Tréves who wrote in the memorial article about Schwartz in October 2003 as follows:<sup>10</sup>

The closest any mathematician of the 1930s ever came to the general definition of a distribution is Sobolev in his articles [Sobolev, 1936] and [Sobolev, 1938]<sup>11</sup> (Leray used to refer to “distributions, invented by my friend Sobolev”). As a matter of fact, Sobolev truly defines the distributions of a given, but arbitrary, finite order  $m$ : as the *continuous linear functionals* on the space  $C_{\text{comp}}^m$  of compactly supported functions of class  $C^m$ . He keeps the integer  $m$  fixed; he never considers the intersection  $C_{\text{comp}}^\infty$  of the spaces  $C_{\text{comp}}^m$  for all  $m$ . This is all the more surprising, since he proves that  $C_{\text{comp}}^{m+1}$  is dense in  $C_{\text{comp}}^m$  by the Wiener procedure of convolving functions  $f \in C_{\text{comp}}^m$  with a sequence of functions belonging to  $C_{\text{comp}}^\infty$ ! In connection with this apparent blindness to the possible role of mentioned to Henri Cartan his inclination to use the elements of  $C_{\text{comp}}^\infty$  as test functions, Cartan tried to dissuade him: “They are too freakish (*trop monstrueuses*).”

Using transposition, Sobolev defines the multiplication of the functionals belonging to  $C_{\text{comp}}^m$  by the functions belonging to  $C^m$  and the differentiation of those functionals:  $d/dx$  maps  $(C_{\text{comp}}^m)^*$  into  $(C_{\text{comp}}^{m+1})^*$ . But again there is no mention of Dirac  $\delta(x)$  nor of convolution, and no link is made with the Fourier transform. He limits himself to applying his new approach to reformulating and solving the Cauchy problem for linear hyperbolic equations. And he does not try to build on his remarkable discoveries. Only after the war does he invent the Sobolev spaces  $H^m$  and then only for integers  $m \geq 0$ . Needless to say, Schwartz had not read Sobolev’s articles, what with military service and a world war (and Western mathematicians’ ignorance of the works of their Soviet colleagues). There is no doubt that knowing those articles would have spared him months of anxious uncertainty.

F. Tréves should be honored for drifting aside from the practice of evaluating publications from what they lack when he wrote somewhat later about that which made the name of Schwartz immortal:<sup>12</sup>

Granted that Schwartz might have been replaceable as the inventor of distributions, what can still be regarded as his greatest contributions to their theory? This writer can mention at least two that will endure: (1) deciding that the Schwartz space  $\mathcal{S}$  of rapidly decaying functions at infinity and its dual  $\mathcal{S}'$  are the “right” framework for Fourier analysis, (2) the Schwartz kernel theorem.

The Tréves opinion coincides practically verbatim with the narration of Schwartz in his autobiography published firstly in 1997. Moreover, Schwartz had even remarked there about Sobolev that<sup>13</sup>

he did not develop his theory in view of general applications, but with a precise goal: he wanted to define the generalized solution of a partial differential equation with a second term and initial conditions. He includes the initial conditions in the second term in the form of functionals on the boundary and obtains in

<sup>10</sup>Cp. [15, p. 1076].

<sup>11</sup>These are references to the articles in *Sbornik* [2, 3].

<sup>12</sup>Ibid., p. 1077.

<sup>13</sup>Cp. [11, p. 222].

this way a remarkable theorem on second order hyperbolic partial differential equations. Even today this remains one of the most beautiful applications of the theory of distributions, and he found it in a rigorous manner. The astounding thing is that he stopped at this point. His 1936 article, written in French, is entitled “Nouvelle méthode à résoudre de problème de Cauchy pour les équations linéaires hyperboliques normales.” After this article, he did nothing further in this fertile direction. In other words, Sobolev himself did not fully understand the importance of his discovery.

It is impossible to agree with these opinions. Rather strange is to read about the absence of any mention of the Dirac delta-function among the generalized functions of Sobolev, since  $\delta$  obviously belongs to each of the spaces  $(C_{\text{comp}}^m)^*$ .

Disappointing is the total neglect of the classical treatise of Sobolev [6] which was a deskbook of many specialists in functional analysis and partial differential equations for decades.<sup>14</sup> Finally, Schwartz was not recruited in 1997 and did not participate in a ‘world war’. Therefore, there were some other reasons for him to neglect the Sobolev book [8] which contains the principally new Sobolev based his pioneering results in numerical integration on developing the theory of the Fourier transform of distributions which was created by Schwartz.

Prudent in the appraisals, exceptionally tactful, and modest in his ripe years, Sobolev always abstained from any bit of details of the history of distribution theory neither in private conversations nor in his numerous writings. The opinion that he decided worthy to be left to the future generations about this matter transpires in his concise comments on the origins of distribution theory in his book [8, Ch. 8] which was printed in 1974:

The generalized functions are “ideal elements” that complete the classical function spaces in much the same way as the real numbers complete the set of rationals.

In this chapter we concisely present the theory of these functions which we need in the sequel. We will follow the way of presentation close to that which was firstly used by the author in 1935 in [16].<sup>15</sup> The theory of generalized functions was further developed by L. Schwartz [21] who has in particular considered and studied the Fourier transform of a generalized function.<sup>16</sup>

Historically, the generalized function had appeared explicitly in the studies in theoretical physics as well as in the works of J. Hadamard, M. Riesz, S. Bochner, et al.

Therefore, we can agree only in part with the following statement by Schwartz [11, p. 236]:

Sobolev and I and all the others who came before us were influenced by our time, our environment and our own previous work. This makes it less glorious, but since we were both ignorant of the work of many other people, we still had to develop plenty of originality.

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<sup>14</sup>Published in 1950 by Leningrad State University, reprinted in 1962 by the Siberian Division of the Academy of Sciences of the USSR in Novosibirsk, and translated into English by the American Mathematical Society in 1963. The third Russian edition was printed by the Nauka Publishers in 1988.

<sup>15</sup>This is a reference to the article of 1936 in *Sbornik* [3].

<sup>16</sup>Cp. [25, p. 355]. This is a curious misprint: the correct reference to Schwartz’s two-volume set should be [47].

Most mathematicians agree that Israel Gelfand could be ranked as the best arbiter in distribution theory. The series *Generalized Functions* written by him and his students was started in the mid 1950s and remains one of the heights of the world mathematical literature, the encyclopedia of distribution theory. In the preface to the first edition of the first volume of this series, Gelfand wrote:<sup>17</sup>

It was S. L. Sobolev who introduced generalized functions in explicit and now generally accepted form in 1936. . . . The monograph of Schwartz *Théorie des Distributions* appeared in 1950–1951. In this book Schwartz systemized the theory of generalized functions, interconnected all previous approaches, laid the theory of topological linear spaces in the foundations of the theory of generalized functions, and obtain a number of essential and far-reaching results. After the publication of *Théorie des Distributions*, the generalized functions won exceptionally swift and wide popularity just in two or three years.

This is an accurate and just statement. We may agree with it.

## 5. CLASSICISM AND ROMANTICISM

Pondering over the fates of Sobolev and Schwartz, it is impossible to obviate the problem of polarization of the opinions about the mathematical discovery of these scholars. The hope is naive that this problem will ever received a simple and definitive answer that satisfies and convinces everyone. It suffices to consider the available experience that concerns other famous pairs of mathematicians whose fates and contributions raise the quandaries that sometimes lasted for centuries and resulted in the fierce clashes of opinions up to the present day. The sources of these phenomena seem of a rather universal provenance that is not concealed in the particular personalities but resides most probably in the nature of mathematical creativity.

Using quite a risky analogy, we may say that mathematics has some features associated with the trends of artistry which are referred to traditionally as classicism and romanticism. It is hard to fail discerning the classic lineaments of the Hellenistic tradition in the writings of Euclid, Newton, Bolyai, Hilbert, and Bourbaki. It is impossible to fail to respond to the accords of the romantic anthem of the human genius which sound in the pages of the writings of Diophant, Leibniz, Lobachevskiĭ, Poincaré, and Arnold.

The magnificent examples of mathematical classicism and romanticism glare from the creative contributions of Sobolev and Schwartz. These giants and their achievements will remain with us for ever.

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<sup>17</sup>Cp. [16].

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